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COMMENT

Directed self-avoiding polygons

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Abstract. The polygon exponent h associated with directed self-avoiding walks in two dimensions has been obtained by exact enumeration and series extrapolation methods. The value of h is found to be -1.40 ± 0.05 and the connectivity constant associated with such polygons is seen to be less than that of the corresponding walks.

Self-avoiding walks (SAW) form a subset of random walks in which no walk crosses one site more than once. Self-avoiding polygons (SAP) are those SAW whose initial and final positions are separated by nearest-neighbour distances.

The statistics of these SAP was studied because of their appearance in several physical problems. For example, the critical temperature of a two-dimensional Ising model was estimated, modelling the boundaries of up (or down) spin domains as SAP (Peierls 1936). The total number of distinct SAW (G_N) and of SAP (C_N) can be fitted to scaling forms $G_N \sim \mu_w^N N^{\nu-1}$ and $C_N \sim \mu_p^N N^h$ for large values of the walk lengths (N). Hammersley (1961) rigorously showed that, for ordinary SAW, $\mu_p = \mu_w$ (which also follows from the n-vector model in the limit $n \to 0$ (de Gennes 1979) as the specific heat and the susceptibility of this system diverge at the same critical temperature). The polygon exponent h obeys a scaling relation $h = -d\nu$ where ν is the average size exponent of ordinary SAW on a d-dimensional lattice. Recently, SAP statistics has been investigated for constrained SAW (Manna 1985a): two-choice SAP are seen to obey ordinary SAP statistics whereas spiral SAP follow a new critical behaviour.

Directed self-avoiding walks (DSAW) form a subset of SAW in which stepping along certain specified lattice directions is forbidden (Chakrabarti and Manna 1983). The critical behaviour (leading 'singularity') of these walks is mean-field-like and is independent of the dimension (Cardy 1983, Redner and Majid 1983, Blöte and Hilhorst 1983, Szpilka 1983) as the effects of excluded volume are suppressed by a major directed term. There are variants of DSAW models in two dimensions. In some models (with stepping forbidden in one or two lattice directions on a square lattice), polygon formation is not possible, but in some other models polygon formation is possible and weak excluded volume effects are present (e.g. in five-choice SAW on a triangular lattice (Redner and Majid 1983)). Recently, the effect of several anisotropic constraints on self-avoiding walk statistics has been studied (Manna 1985b). All these anisotropic SAW behave as DSAW in critical behaviour. In these models polygon formation is also possible. Although the excluded volume effects in these models do not show up in the leading behaviours of the walk statistics, they appear to manifest themselves in the polygon statistics. Here we consider two anisotropically constrained SAW (their critical behaviour is the same as DSAW) and one directed SAW model in two dimensions

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and calculate the polygon exponent h for each of them. The model walks are the following (walk statistics belong to the directed SAW universality class in each case).

(i) On a square lattice, a step along a particular lattice direction does not allow its next step to be in the same direction (walk statistics discussed in Manna (1985b)).

(ii) On a triangular lattice, a step along a particular lattice direction does not allow its next step to be in the same direction.

(iii) On a triangular lattice, a particular lattice direction is totally forbidden for stepping (walk statistics discussed in Redner and Majid (1983)).

In these three models, the first two models are not directed SAW models from their definition. However, in these models choice of stepping is anisotropic with respect to a particular lattice direction. Recently it has been shown that any finite anisotropy (however small) imposed on the isotropic SAW, leads to DSAW critical behaviour (Manna and Chakrabarti 1984). Therefore, it is expected that models (i) and (ii) will also behave as DSAW. In fact, this has already been shown for model (i) (Manna 1985b), and model (ii) is nothing but model (i) defined on a triangular lattice. We thus expect that model (ii) will also belong to the DSAW universality class.

We have enumerated exactly all the distinct SAP (C_N) for a particular perimeter length N. These simulation data have been given in table 1. We estimated the connectivity constant μ and the polygon exponent h in the following way:

$$\mu_N = \left(\frac{(N+n)^x C_{N+n}}{N^x C_N}\right)^{1/n} \tag{1}$$

$$h_N = \{\ln(\mu_N|_{x=0}/\mu)/\ln[(N+n)/N]\}n. \tag{2}$$

Equation (1) has been adopted from Manna (1985a). In this equation, n is introduced and its value is suitably chosen to minimise oscillations in μ_N values and for better

Table 1. Exact enumeration data for the total number of distinct directed SAP: C_N^A, C_N^B and C_N^C correspond to models (i), (ii) and (iii) respectively.

N	C_N^A	C_N^B	C_N^C
1			
2			
3		12	6
4	8	24	8
5		52	20
6	14	150	36
7		466	112
8	54	1 482	272
9		4 988	756
10	222	16 992	2 120
11		58 828	5 896
12	940	206 486	17 064
13		731 972	48 932
14	4 190	2618 612	142 996
15			419 520
16	19 248		
17			
18	90 944		
19			
20	436 786		

estimation of μ after extrapolation. The actual value of μ (μ_N for $N \to \infty$) does not depend on n. The quantity x is analogous to the negative of the exponent h and the actual value of μ does not depend on x. This is introduced to obtain different sets of μ_N values. The choice of x helps to tune the value of the slope of the μ_N against $1/N$ curve (from positive to negative) and thus reduce the error in the estimation of extrapolated μ.

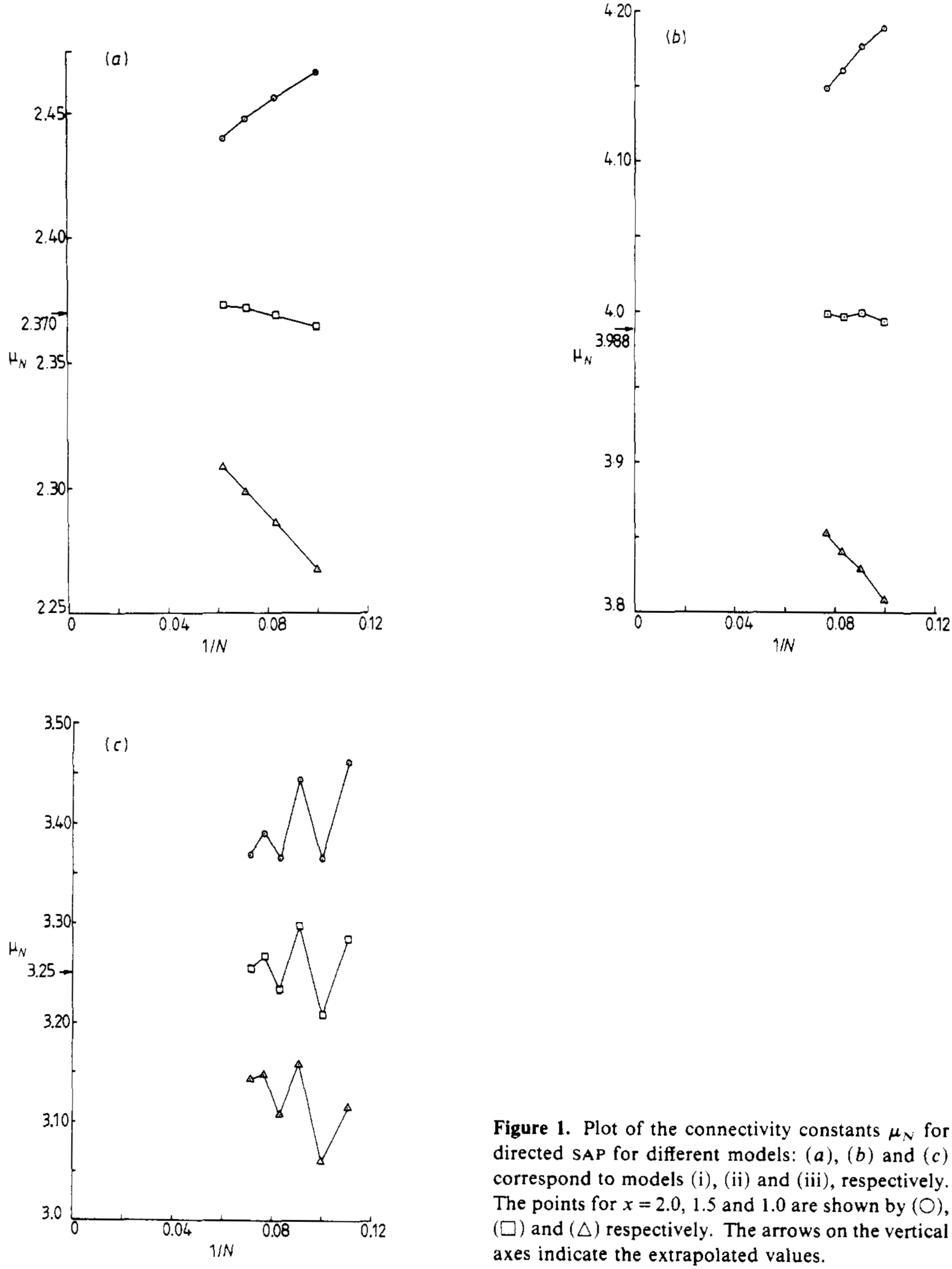
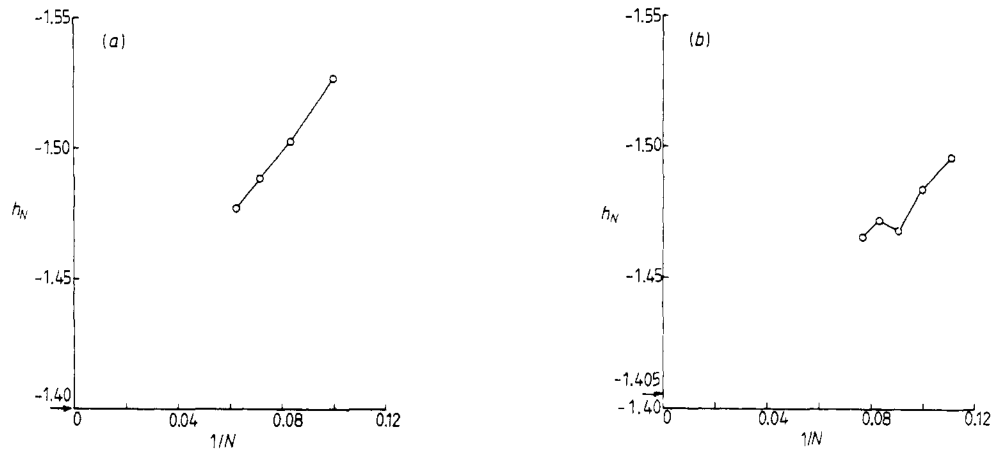


Figure 1. Plot of the connectivity constants μ_N for directed SAP for different models: (*a*), (*b*) and (*c*) correspond to models (i), (ii) and (iii), respectively. The points for $x = 2.0$, 1.5 and 1.0 are shown by (○), (□) and (△) respectively. The arrows on the vertical axes indicate the extrapolated values.

Using equation (1) we have calculated μ_N with $x = 1.0$, 1.5 and 2.0 for all three models, extrapolated them and obtained μ_p of the polygons (see figure 1). Substituting these μ_p values in (2) we obtained h_N series. Extrapolation of these quantities gives h (see figure 2). For the first model we used $n = 4$ (to minimise oscillations) and for the other models $n = 1$. We obtained 2.370 ± 0.005, 3.988 ± 0.005 and 3.25 ± 0.05 for the connectivity constants (μ_p) of the polygons of models (i), (ii) and (iii) respectively. The exponent h was found to be -1.40 ± 0.01, -1.41 ± 0.01 and -1.37 ± 0.05 for models (i), (ii) and (iii), indicating $h \simeq -1.40$ for all the models in $d = 2$. We compare these values of the connectivity constants for polygons to the corresponding values for walks. We have two values of the connectivity constants μ_w for walks of two models, namely for walks of types (i) and (iii). They are 2.535 ± 0.005 (Manna 1985b) and 3.86 ± 0.01 (Manna 1986), respectively. It is, thus, to be noted that (at least for these two cases) the connectivity constants for walks are greater than those for polygons and the magnitude of the directed SAP exponent (h) is found to be very close to that for ordinary SAP ($h = -1.5$ (Nienhuis 1982)). It may be mentioned that, unlike the walk statistics, the polygon statistics of directed SAW are essentially governed by the excluded volume effect (where both the advanced and the retarded parts of the bare propagator enter the calculation) which makes any extension of the complex n-component ($n \to 0$) field model for the directed polygons difficult and non-trivial, although such a treatment would indicate $\mu_p < \mu_w$ (Cardy 1986).



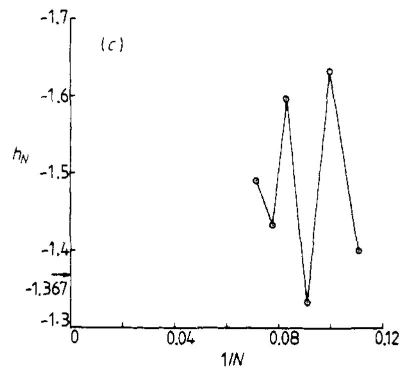


Figure 2. Plot of the polygon exponent h_N for directed SAP for different models: (*a*), (*b*) and (*c*) correspond to models (i), (ii) and (iii), respectively. The arrows on the vertical axes indicate the extrapolated values.

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